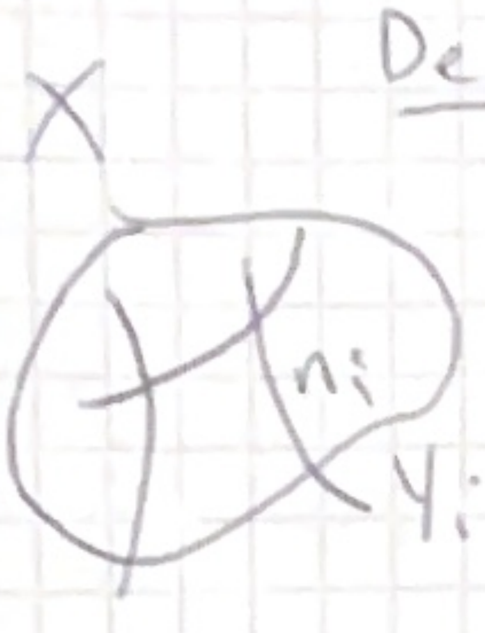
Divisors and line bundles

X ~~smth~~ ns connected variety $/k$.
($\Rightarrow X$ integral w/ generic pt η)
(most constructions work in greater generality)

Weil divisors

What is a Weil divisor?



Def · A prime divisor $Y \subseteq X$ is a closed integral subscheme of codim 1.

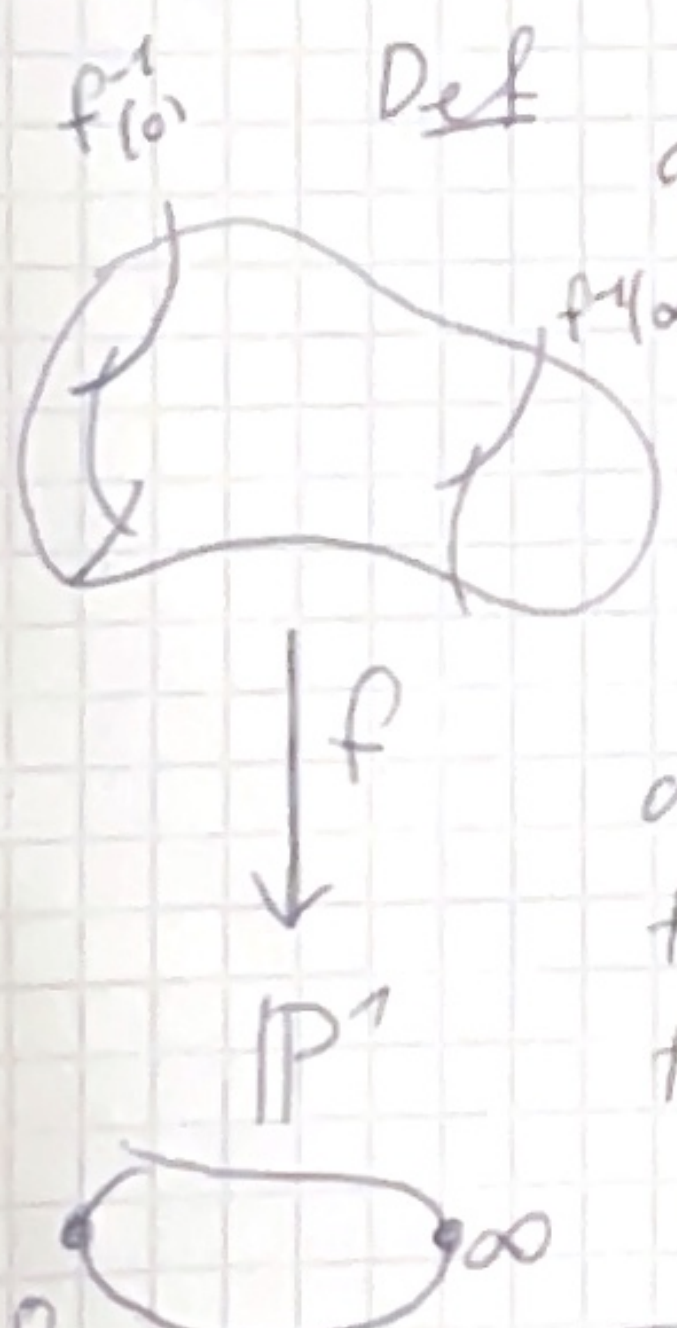
· A (Weil) divisor $D = \sum n_i Y_i \in \mathrm{Div}(X) = \bigoplus_{\substack{Y \subset X \\ \text{prime divisor}}} \mathbb{Z} Y$

· D is effective if $n_i \geq 0 \quad \forall i$.

· $\deg(D) = \sum n_i$

Let $Y \subseteq X$ prime divisor
X ns $\Rightarrow \mathcal{O}_{X,Y}$ regular local ring of dim 1
i.e. a DVR w/ fraction field $K(X)$
and valuation

$$\mathrm{ord}_Y : K(X)^\times \to \mathbb{Z}$$
$$f = ut^n \mapsto n$$
$$u \in \mathcal{O}_{X,Y}^\times$$
$$t \in \mathfrak{m}_Y \text{ uniformizer.}$$



Def divisor of zeros and poles homomorphism

$$\mathrm{div} : K(X)^\times \to \mathrm{Div}(X)$$
$$f \mapsto \sum_{\substack{Y \subseteq X \\ \text{prime}}} \mathrm{ord}_Y(f)\, Y$$

divisors of the form $\mathrm{div}(f)$ are called principal
two divisors ~~on~~ D, D' are linearly equivalent if they differ by a principal divisor i.e. $\exists f \in K(X)^\times$ s.t. $\mathrm{div}(f) = D - D'$

The divisor class group $\mathrm{Cl}(X) = \mathrm{Div}(X) / \underbrace{\mathrm{im}(\mathrm{div})}_{=: \mathrm{PDiv}(X)}$

Two facts from commutative algebra:

~~Prop~~ Fact ~~1~~ A noetherian domain TFAE

(1) A is a UFD

(2) $\mathrm{Spec}(A)$ normal and $\mathrm{Cl}(\mathrm{Spec}(A)) = 0$

(3) every prime ideal of height 1 is principal.

Fact If R is a regular local ring, then R is a UFD.

⇝ for us $\mathrm{Cl}(X)$ is a global / geometric invariant (in general, it also contains 'arithmetic' information)

Cartier divisors — in exercises saw $K(X)$ of integral schemes & properties

$\mathcal{K}_X$: constant ~~sh~~ $\mathcal{O}_X$-module valued in $K(X)$

$\cup|$

$\mathcal{K}_X^\times$: constant sheaf of abelian grps — valued in $K(X)^\times$

Have SES (of sheaves of abelian groups written multiplicatively)

$$1 \to \mathcal{O}_X^\times \longrightarrow \mathcal{K}_X^\times \longrightarrow \mathcal{K}_X^\times / \mathcal{O}_X^\times \to 1$$

Def A Cartier divisor $\in \Gamma(X, \mathcal{K}_X^\times / \mathcal{O}_X^\times)$

$$\Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times) = \left\{ (U_i, f_i) \;\middle|\; \begin{array}{l} X = \bigcup U_i \\ f_i \in K(X)^\times \text{ st } f_i/f_j \in \mathcal{O}_X(U_{ij})^\times \end{array} \right\} \Big/ \sim$$

$$1 \to \Gamma(X, \mathcal{O}_X^\times) \to \underset{\substack{\| \\ K(X)^\times}}{\Gamma(X, \mathcal{K}_X^\times)} \xrightarrow{\text{cart}} \Gamma(X, \mathcal{K}_X^\times / \mathcal{O}_X^\times) \to \cdots$$

Cartier divisors in the image of cart are called principal

For all $Y \subset X$ prime $\operatorname{ord}_Y|_{\mathcal{O}_X^\times} = 0$ $\Rightarrow$

$$\Rightarrow \begin{tikzcd} K(X)^\times \arrow[r] \arrow[dr, "\operatorname{ord}_Y"'] & \Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times) \arrow[d, "\operatorname{ord}_Y"] \\ & \mathbb{Z} \end{tikzcd}$$

$$\leadsto \begin{tikzcd} K(X)^\times \arrow[r] \arrow[dr, "\operatorname{div}"'] & \Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times) \arrow[d, "\operatorname{div}"] \\ & \operatorname{Div}(X) \end{tikzcd}$$

<u>Proposition</u> X ns connected var$/k$
$\operatorname{div} : \Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times) \to \operatorname{Div}(X)$
is an isomorphism

<u>Proof</u> For all $x \in X$ $\mathcal{O}_{X,x}$ regular local domain
$\Rightarrow$ UFD

For all $Y \subseteq X$ prime
$Y|_{\operatorname{Spec}(\mathcal{O}_{X,x})} \subset \operatorname{Spec}(\mathcal{O}_{X,x})$
is a closed subvariety cut out by a height one ~~p~~ <u>principal</u> prime ideal.
$\mathfrak{p}_{Y,x} = (f_{Y,x})$

Choose open cover $X = \bigcup U_{Y,i}$
and $f_{Y,i} \in \mathcal{O}_X(U_{Y,i})$ s.t. $Y|_{U_{Y,i}} = V(f_{Y,i})$

The map $\operatorname{Div}(X) \to \Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$
$Y \mapsto (U_{Y,i}, f_{Y,i})$ ← class does not depend on choice $U_{Y,i}, f_{Y_i}$
yields an inverse to div.

<u>Exercise</u>: $\operatorname{div}(U_i, f_i)$ is effective iff $f_i \in \mathcal{O}_{U_i}^\times$

Line bundles

Def A line bundle (a.k.a. invertible sheaf) $\mathcal{L}$ on X is a sheaf of $\mathcal{O}_X$-modules which is (Zariski-) locally isomorphic to $\mathcal{O}_X$, i.e., $\exists$ open cover $X = \bigcup_i U_i$ and isomorphism $\mathcal{O}_{U_i} \cong \mathcal{L}|_{U_i}$

$$(\mathcal{O}_X(-), s_-) : \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \to \left\{ (\mathcal{L}, s) \;\middle|\; \begin{array}{l} \mathcal{L} \text{ line bundle} \\ s \in \mathcal{L}_\eta \setminus \{0\} \end{array} \right\} / \sim$$

$$D = (U_i, f_i) \mapsto \left(\mathcal{O}_X(D)(U_i) := f_i^{-1} \mathcal{O}_X(U_i) \subseteq K(X)^*, \; s_D := 1 \right)$$

$$\mathcal{O}_X(D)(U_i) = \{ f \in K(X)^* \mid \operatorname{div}(f|_U) + D|_U \geq 0 \} \cup \{0\}$$

Prop $(\mathcal{O}_X(-), s_-)$ is an isomorphism and induces an isomorphism

$$\mathrm{Cl}(X) \cong \mathrm{CaCl}(X) \longrightarrow \mathrm{Pic}(X)$$

Proof Inverse construction: $(\mathcal{L}, s) \in$ RHS

Pick a trivialisation $X = \bigcup U_i$

$$\varphi_i : \mathcal{L}|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}$$

$$\mathrm{cart}(\mathcal{L}, s) := (U_i, \varphi_i(s|_{U_i}))$$ $\square$

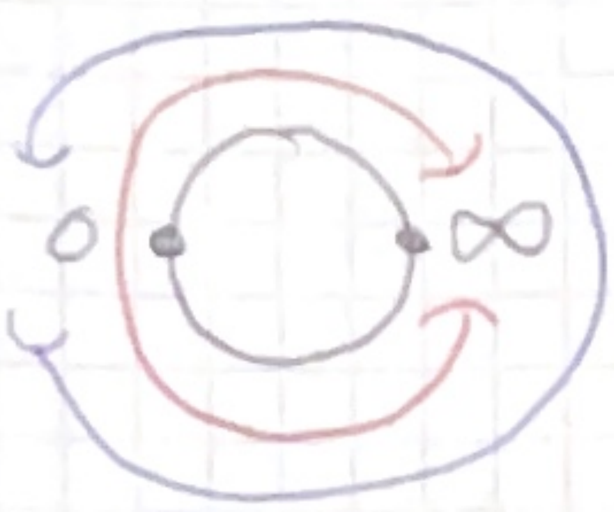
Summary:

$$\begin{tikzcd}
\{(\mathcal{L}, s)\}/\sim \arrow[d] & \Gamma(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \arrow[l, "\mathcal{O}_X(-), s_-"', "\cong"] \arrow[r, "\operatorname{div}", "\cong"'] \arrow[d] & \mathrm{Div}(X) \arrow[d] \\
\mathrm{Pic}(X) & \mathrm{CaCl}(X) \arrow[l, "\cong"'] \arrow[r, "\cong"'] & \mathrm{Cl}(X)
\end{tikzcd}$$

$\Rightarrow$ set of linearly equivalent divisors to D is parameterised by $\Gamma(X, \mathcal{O}_X(D)) \setminus \{0\} / k^\times$

[Hart, II.7] $\rightsquigarrow$ motivates definition of linear systems (after example)

Example: $D = [0] - 2[\infty] \in \mathrm{Div}(\mathbb{P}^1)$



$\mathbb{P}^1 = \underline{U_0} \cup \underline{U_1} = \underline{\mathrm{Spec}(k[s])} \cup \mathrm{Spec}(\underline{k[t]})$

$[0]|_{U_0} = \emptyset \qquad [0]|_{U_1} = \mathrm{Spec}(k[t]/(t))$

$[\infty]|_{U_0} = \mathrm{Spec}(k[s]/(s)) \qquad [\infty]|_{U_1} = \emptyset$

$[0] \longleftrightarrow (U_0, 1) \quad (U_1, t)$

$[\infty] \longleftrightarrow (U_0, s) \quad (U_1, 1)$

$\Rightarrow \quad D \longleftrightarrow (U_0, s^{-2}) \quad (U_1, t)$

$\mathcal{O}_X(D)|_{U_1} = t\,k[t] \quad , \quad \mathcal{O}_X(D)|_{U_0} = s^{-2} k[s]$

$\mathcal{O}_X(D)|_{U_{10}} = t\,k[t^{\pm 1}] \xrightarrow{\cong} \mathcal{O}_X(D)|_{U_{01}} = s^{-2} k[s^{\pm 1}]$

$f(t) \longmapsto \frac{s^{-2}}{t} f(s^{-1}) = s^{-1} f(s^{-1})$

$t^{-1} g(t) = \frac{t}{s^2} g(t^{-1}) \longmapsfrom g(s)$

Functoriality for finite ~~maps~~ do

f

Linear Systems

Def The *complete linear system* of a divisor D is

$$|D| := \Gamma(X, \mathcal{O}_X(D)) - \{0\}/k^\times \text{ "=" } \mathbb{P}(\Gamma(X, \mathcal{O}_X(D)))(k$$

A linear system of D is a liner subspace $\delta \subseteq |D|$

A point $p \in X$ is a *base pt* for δ if $p \in \operatorname{supp}(D')$ for all $D' \in \delta$

$$\operatorname{supp}(D') = \bigcup_i Y_i' \qquad D' = \sum n_i' Y_i'$$

Relation to maps to $\mathbb{P}^n$

Def $\mathcal{O}_{\mathbb{P}^n}(1)$ is the line bundle on $\mathbb{P}^n$ with global sections $\Gamma(\mathbb{P}^n, \mathcal{O}(1)) = \underbrace{k[x_0, \dots, x_n]_1}_{\text{homogenous linear fcts of deg 1}}$

Should always think of $\mathbb{P}^n$ together w/ $\mathcal{O}_{\mathbb{P}^n}(1)$

$x_0, \dots, x_n \in \Gamma(\mathbb{P}^n, \mathcal{O}(1))$ globally generate $\mathcal{O}_{\mathbb{P}^n}(1)$:

i.e. $\mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} \xrightarrow{\oplus x_i} \mathcal{O}_{\mathbb{P}^n}(1)$ is surjective.

$\Rightarrow$ For all $X \xrightarrow{\pi} \mathbb{P}^n$ $\pi^* \mathcal{O}_{\mathbb{P}}(1)$ is globally generated by $\pi^* x_0, \dots, \pi^* x_n$

Conversely, if $(\mathcal{L}, s_0, \dots, s_n)$ is a line bundle with globally generating sections $s_0, \dots, s_n$ (a.k.a. base pt free)

can define a morphism

$$X \xrightarrow{\pi} \mathbb{P}^n \text{ "=" } \mathbb{P}(\Gamma(X, \mathcal{L}))$$
$$p \longmapsto [s_0(p), \dots, s_n(p)]$$

s.t. $\pi^* \mathcal{O}_{\mathbb{P}^n}(1) = \mathcal{L}$ and $\pi^* x_i = s_i$

Theorem

$$\mathrm{Hom}_{Sch/k}(X, \mathbb{P}^n) = \left\{ (\mathcal{L}, s_0, \ldots, s_n) \;\middle|\; \begin{array}{l} \mathcal{L} \text{ line bundle on } X \\ s_0, \ldots, s_n \text{ globally generating} \\ \text{sections of } \mathcal{L} \end{array} \right\}$$

$$\underset{(X \text{ smooth})}{=} \left\{ (\delta, s_0, \ldots, s_n) \;\middle|\; \begin{array}{l} \delta \text{ linear system of dim } n \\ \cancel{\text{basis of}} \\ s_0, \ldots, s_n \in \delta \text{ base pt free.} \end{array} \right\}$$

Def X/k variety is projective if there is a closed embedding $X \hookrightarrow \mathbb{P}^n$
equivalently
- X has a very ample line bundle
- X has an ample line bundle

Cor $\mathbb{P}^n$ is proper

Pf idea: Val. Crit. Diag:

$$\begin{tikzcd} \mathrm{Spec}(K) \arrow[r, "\pi"] \arrow[d] & \mathbb{P}^n_k \\ \mathrm{Spec}(R) \arrow[ur, dashed] & \end{tikzcd}$$

$$\pi \longleftrightarrow (\mathcal{L}, s_0, \ldots, s_n) \quad \text{on } \mathrm{Spec}(K)$$
$$\wr\!\wr$$
$$\left(K, \frac{f_0}{g_0}, \ldots, \frac{f_n}{g_n}\right) \quad f_i, g_i \in R$$

$$\tilde{\pi} \longleftrightarrow \left(\frac{1}{g_0 \cdots g_n} R, \frac{f_0}{g_0}, \ldots, \frac{f_n}{g_n} \right)$$ □

Cor Every projective variety is proper

Pf
$$\begin{tikzcd} X \arrow[r, hook, "\text{proper}"] \arrow[dr] & \mathbb{P}^n \arrow[d, "\text{proper}"] \\ & k \end{tikzcd}$$ □